J. Number Theory,
128 (2008), p. 2214-2230.

Upper bounds for the order of an additive basis obtained by removing a finite subset of a given basis

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Abstract

Let A be an additive basis of order h and X be a finite nonempty subset of A such that the set $A \setminus X$ is still a basis. In this article, we give several upper bounds for the order of $A \setminus X$ in function of the order h of A and some parameters related to X and A. If the parameter in question is the cardinality of X, Nathanson and Nash already obtained some of such upper bounds, which can be seen as polynomials in h with degree $(|X| + 1)$. Here, by taking instead of the cardinality of X the parameter defined by $d := \frac{\mathrm{diam}(X)}{\gcd\{x-y \mid x,y\in X\}}$, we show that the order of $A \setminus X$ is bounded above by $(\frac{h(h+3)}{2} + d\frac{h(h-1)(h+4)}{6})$. As a consequence, we deduce that if X is an arithmetic progression of length ≥ 3, then the upper bounds of Nathanson and Nash are considerably improved. Further, by considering more complex parameters related to both X and A, we get upper bounds which are polynomials in h with degree only 2.

MSC: 11B13

Keywords: Additive basis; Kneser's theorem.

1 Introduction

An additive basis (or simply a basis) is a subset A of $\mathbb{Z}$, having a finite intersection with $\mathbb{Z}^-$ and for which there exists a natural number h such that any sufficiently large positive integer can be written as a sum of h elements of A. The smaller number h satisfying this property is called "the order of the basis A" and we note it $\mathrm{G}(A)$. If A is a basis of order h and X is a finite nonempty subset of A such that $A \setminus X$ is still a basis, the problem dealt with here is to find upper bounds for the order of $A \setminus X$ in function of the order h of A and parameters related to X (resp. X and A). The particular case when X contains only one element, say $X = \{x\}$, was studied for the first time by Erdös and Graham [1]. These two last authors showed that $\mathrm{G}(A \setminus \{x\}) \leq \frac{5}{4}h^2 + \frac{1}{2}h \log h + 2h$. After hem, several works followed in order to improve this estimate: In his Thesis, by using Kneser's theorem (see e.g. [5] or [4]), Grekos [2] improved the previous estimate to

$G(A\setminus\{x\}) \leq h^2+h$. By still using Kneser's theorem but in a more judicious way, Nash [7] improved the estimate of Grekos to $G(A \setminus \{x\}) \leq \frac{1}{2}(h^2+3h)$. Finally, by combining Kneser's theorem with some new additive methods, Plagne [10] obtained the refined estimate $G(A\setminus\{x\}) \leq \frac{h(h+1)}{2} + \lceil \frac{h-1}{3} \rceil$, which is best known till now. Plagne conjectured that $G(A \setminus \{x\}) \leq \frac{h(h+1)}{2} + 1$, but this has not yet been proved. Notice also that the optimality of such estimates was discussed by different authors (see e.g. [1], [2], [3], [10]).

The general case of the problem was studied by Nathanson and Nash (see e.g. [9], [6], [8] and [7]). For $h, k \in \mathbb{N}$, these two authors noted $G_k(h)$ the maximum of all the natural numbers $G(A \setminus X)$, where A is an additive basis of order h and X is a subset of A with cardinality k such that $A \setminus X$ is still a basis. In [8], they proved that $G_k(h)$ has order of magnitude h^{k+1}. Indeed, they showed that

$$\left(\frac{h}{k+1}\right)^{k+1} + O(h^k) \leq G_k(h) \leq \frac{2}{k!} h^{k+1} + O(h^k)$$

(see Theorem 4 of [8]).
Since then, the above bounds of $G_k(h)$ were improved. In [11], Xing-de Jia showed that

$$G_k(h) \geq \frac{4}{3}\left(\frac{h}{k+1}\right)^{k+1} + O(h^k)$$

and in [7], Nash obtained the following

Theorem 1.1 ([7], Proposition 3 simplified) *Let A be a basis and X be a finite subset of A such that $A \setminus X$ is still a basis. Then, noting h the order of A and k the cardinality of X, we have:*

$$G(A \setminus X) \leq (h+1)\binom{h+k-1}{k} - k\binom{h+k-1}{k+1}.$$

Actually, the original estimate of Nash (Proposition 3 of [7]) is that $G(A \setminus X) \leq \binom{h+k-1}{k} + \sum_{i=0}^{h-1} \binom{k+i-1}{i}(h-i)$. But we can simplify this by remarking that for all $i \in \mathbb{N}$, we have:

$$\binom{k+i-1}{i} = \binom{k+i}{i} - \binom{k+i-1}{i-1}$$

and

$$i\binom{k+i-1}{i} = k\binom{k+i-1}{i-1} = k\left\{\binom{k+i}{i-1} - \binom{k+i-1}{i-2}\right\}.$$

Consequently, we have:

$$\sum_{i=0}^{h-1}\binom{k+i-1}{i}(h-i) \;=\; h\sum_{i=0}^{h-1}\binom{k+i-1}{i} - \sum_{i=0}^{h-1} i\binom{k+i-1}{i}$$

$$
\begin{aligned}
&= h\sum_{i=0}^{h-1}\left\{\binom{k+i}{i}-\binom{k+i-1}{i-1}\right\}-k\sum_{i=0}^{h-1}\left\{\binom{k+i}{i-1}-\binom{k+i-1}{i-2}\right\}\\
&= h\binom{h+k-1}{h-1}-k\binom{h+k-1}{h-2}\\
&= h\binom{h+k-1}{k}-k\binom{h+k-1}{k+1},
\end{aligned}
$$

which leads to the estimate of Theorem 1.1.

In Theorem 1.1, the upper bound of $\mathrm{G}(A\backslash X)$ is easily seen to be a polynomial in h with leading term $\frac{h^{k+1}}{(k+1)!}$, thus with degree $(k+1)$. In this paper, we show that it is even possible to bound from above $\mathrm{G}(A \setminus X)$ by a polynomial in h with degree constant (3 or 2) but with coefficients depend on a new parameter other the cardinality of X. By setting

$$d := \frac{\mathrm{diam}(X)}{\delta(X)},$$

where $\mathrm{diam}(X)$ denotes the usual diameter of X and $\delta(X) := \gcd\{x-y \mid x, y \in X\}$, we show that

$$\mathrm{G}(A \setminus X) \leq \frac{h(h+3)}{2} + d\frac{h(h-1)(h+4)}{6} \qquad \text{(see Theorem 4.1).}$$

Also, by setting

$$\eta := \min_{\substack{a,b\in A\setminus X, a\neq b\\ |a-b|\geq \mathrm{diam}(X)}} |a-b|,$$

we show that

$$\mathrm{G}(A \setminus X) \leq \eta(h^2-1) + h + 1 \qquad \text{(see Theorem 4.3).}$$

Finally, by setting

$$\mu := \min_{y\in A\setminus X} \mathrm{diam}(X \cup \{y\}),$$

we show that

$$\mathrm{G}(A \setminus X) \leq \frac{h\mu(h\mu+3)}{2} \qquad \text{(see Theorem 4.4).}$$

It must be noted that this last estimate is obtained by an elementary way as a consequence of Nash' theorem while the two first estimates are obtained by applying Kneser's theorem with some differences with [7].

In practice, when h and k are large enough, it often happens that our estimates are better than that of Theorem 1.1. The more interesting corollary is when X is an arithmetic progression: in this case we have $d = k - 1$, implying from our first estimate an improvement of Theorem 1.1.

2 Notations, terminologies and preliminaries

2.1 General notations and elementary properties

(1) If X is a finite set, we let $|X|$ denote the cardinality of X. If in addition $X \subset \mathbb{Z}$ and $X \neq \emptyset$, we let $\mathrm{diam}(X)$ denote the usual diameter of X (that is $\mathrm{diam}(X) := \max_{x,y \in X} |x - y|$) and we let

$$\delta(X) := \gcd\{x - y \mid x, y \in X\}$$

(with the convention $\delta(X) = 1$ if $|X| = 1$).

(2) If B and C are two sets of integers, the notation $B \sim C$ means that the symmetric difference $B \Delta C$ $(= (B \setminus C) \cup (C \setminus B))$ is finite; namely B and C differ just by a finite number of elements.

(3) If $A_1, A_2, \dots, A_n$ $(n \geq 1)$ are nonempty subsets of an abelian group, we write

$$\sum_{i=1}^{n} A_i := \{a_1 + a_2 + \cdots + a_n \mid a_i \in A_i\}.$$

If $A_1 = A_2 = \cdots = A_n \neq \mathbb{Z}$, it is convenient to write the previous set as nA_1; while $n\mathbb{Z}$ stands for the set of the integer multiples of n.

(4) If $U = (u_i)_{i \in \mathbb{N}}$ is a nondecreasing and non-stationary sequence of integers, we write, for all $m \in \mathbb{N}$, $U(m)$ the number of terms of U not exceeding m. (Stress that if U is increasing, then it is just considered as a subset of $\mathbb{Z}$ having a finite intersection with $\mathbb{Z}^-$).
• We call "the lower asymptotic density" of U the quantity defined by

$$\underline{\mathbf{d}}(U) := \liminf_{m \to +\infty} \frac{U(m)}{m} \in [0, +\infty].$$

If U is increasing (so it is a subset of $\mathbb{Z}$ having a finite intersection with $\mathbb{Z}^-$), we clearly have $\underline{\mathbf{d}}(U) \leq 1$.

(5) If $U_1, U_2, \dots, U_n$ $(n \geq 1)$ are nondecreasing and non-stationary sequences of integers, indexed by $\mathbb{N}$, the notation $U_1 \vee U_2 \vee \cdots \vee U_n$ (or $\vee_{i=1}^{n} U_i$) represents the aggregate of the elements of $U_1, \dots, U_n$; each element being counted according to its multiplicity.
• It's clear that for all $m \in \mathbb{N}$, we have: $(U_1 \vee \cdots \vee U_n)(m) = \sum_{i=1}^{n} U_i(m)$. So, it follows that:

$$\underline{\mathbf{d}}(U_1 \vee \cdots \vee U_n) \geq \sum_{i=1}^{n} \underline{\mathbf{d}}(U_i).$$

• Further, if $U_1, \dots, U_n$ are increasing (so they are simply sets), we clearly have:

$$\underline{\mathbf{d}}(U_1 \vee \cdots \vee U_n) \geq \underline{\mathbf{d}}(U_1 \cup \cdots \cup U_n).$$

(6) It is easy to check that if U is a nondecreasing and non-stationary sequence of integers (indexed by $\mathbb{N}$) and $t \in \mathbb{Z}$, then we have:

$$(U+t)(m) = U(m) + O(1).$$

(7) If B is a nonempty set of integers and g is a positive integer, we denote $\frac{B}{g\mathbb{Z}}$ the image of B under the canonical surjection $\mathbb{Z} \to \frac{\mathbb{Z}}{g\mathbb{Z}}$. We also denote $B^{(g)}$ the set of all natural numbers which are congruent modulo g to some element of B; in other words:

$$B^{(g)} := (B + g\mathbb{Z}) \cap \mathbb{N}.$$

• We can easily check that if B and C are two nonempty sets of integers and g is a positive integer, then we have:

$$(B+C)^{(g)} \sim B^{(g)} + C.$$

In particular, if we have $B \sim B^{(g)}$ then we also have $B + C \sim (B+C)^{(g)}$.

2.2 The theorems of Kneser (see [4], Chap 1)

Theorem 2.1 (The first theorem of Kneser)
Let $A_1, A_2, \dots, A_n$ $(n \geq 1)$ be nonempty sets of integers having each one a finite intersection with $\mathbb{Z}^-$. Then either

$$\underline{\mathbf{d}}\left(\sum_{i=1}^{n} A_i\right) \geq \underline{\mathbf{d}}\left(\bigvee_{i=1}^{n} A_i\right) \tag{I}$$

or there exists a positive integer g such that

$$\sum_{i=1}^{n} A_i \sim \left(\sum_{i=1}^{n} A_i\right)^{(g)}. \tag{II}$$

Remarks:
• We call (I) "the first alternative of the first theorem of Kneser" and we call (II) "the second alternative of the first theorem of Kneser".
• The relation (II) implies in particular that the set $\sum_{i=1}^{n} A_i$ is (starting from some element) a finite union of arithmetic progressions with common difference g.

Theorem 2.2 (The second theorem of Kneser)
Let G be a finite abelian group and B and C be two nonempty subsets of G. Then, there exists a subgroup H of G such that

$$B + C = B + C + H$$

and

$$|B+C| \geq |B+H| + |C+H| - |H|.$$

In the applications, we use the second theorem of Kneser in the form given by the corollary below. We first need to define the so-called "a subset not degenerate of an abelian group" and then to give a simple property related to this one.

Definitions:
• If G is an abelian group and B is a subset of G, we say that "B is not degenerate in G" if we have $\mathrm{stab}_G(B) = \{0\}$ (where $\mathrm{stab}_G(B)$ denotes the stabilizer of B in G).
• If B is a set of integers and g is a positive integer, we say that "B is not degenerate modulo g" if $\frac{B}{g\mathbb{Z}}$ is not degenerate in $\frac{\mathbb{Z}}{g\mathbb{Z}}$.

Proposition 2.3 *Let G be an abelian group and B and C be two nonempty subsets of G such that $(B + C)$ is not degenerate in G. Then also B and C are not degenerate in G.*

Proof. This is an immediate consequence of the fact that:
$\mathrm{stab}_G(B) + \mathrm{stab}_G(C) \subset \mathrm{stab}_G(B + C)$. ■

Corollary 2.4 *Let G be a finite abelian group and $B_1, \dots, B_n$ $(n \geq 1)$ be nonempty subsets of G such that $(B_1 + \dots + B_n)$ is not degenerate in G. Then we have*

$$|B_1 + \dots + B_n| \geq |B_1| + \dots + |B_n| - n + 1.$$

Proof. It suffices to show the corollary for $n = 2$. The general case follows by a simple induction on n and by using Proposition 2.3. Suppose $n = 2$. Theorem 2.2 gives a subgroup H of G satisfying the two relations $B_1 + B_2 = B_1 + B_2 + H$ and $|B_1 + B_2| \geq |B_1 + H| + |B_2 + H| - |H|$. The first one implies $H \subset \mathrm{stab}_G(B_1 + B_2) = \{0\}$, so $H = \{0\}$. By replacing this into the second one, we conclude to $|B_1 + B_2| \geq |B_1| + |B_2| - 1$ as required. ■

The following proposition (which is an easy exercise) makes the connection between the first and the second theorem of Kneser:

Proposition 2.5 *Let B be a nonempty set of integers and g be a positive integer. The two following assertions are equivalent:*

(i) *B is not degenerate modulo g*

(ii) *There is no positive integer $m < g$ such that $B^{(m)} = B^{(g)}$.*

Now, let us explain how we use the theorems of Kneser in this paper. We first get sets $A_i = h_i(A \setminus X)$, $i = 0, \dots, n$ such that $\cup_{i=1}^{n}(A_i + \tau_i) \sim \mathbb{N}$ and $\underline{\mathbf{d}}(A_0) > 0$ (where n is a natural number depending on A and X, the h_i's are positive integers depending only on h and such that $h_0 \leq n$ and the τ_i's are integers). We thus have $\underline{\mathbf{d}}(\vee_{i=0}^{n} A_i) > 1$, implying that the first alternative of the first theorem of Kneser cannot hold. Consequently we are in the second alternative of the first theorem of Kneser, namely there exists a positive integer g such that $\sum_{i=0}^{n} A_i \sim (\sum_{i=0}^{n} A_i)^{(g)}$. By choosing g minimal to have this property, we

deduce from Proposition 2.5 that the set $\sum_{i=0}^{n} A_i$ is not degenerate modulo g; in other words the set $\sum_{i=0}^{n} \frac{A_i}{g\mathbb{Z}}$ is not degenerate in the group $\frac{\mathbb{Z}}{g\mathbb{Z}}$. It follows from Proposition 2.3 that also $\sum_{i=1}^{n} \frac{A_i}{g\mathbb{Z}}$ is not degenerate in $\frac{\mathbb{Z}}{g\mathbb{Z}}$. Then by applying Corollary 2.4 for $G = \frac{\mathbb{Z}}{g\mathbb{Z}}$ and $B_i = \frac{A_i}{g\mathbb{Z}}$ $(i = 1, \dots, n)$, we deduce that $\left|\frac{\sum_{i=1}^{n} A_i}{g\mathbb{Z}}\right| \geq \sum_{i=1}^{n} \left|\frac{A_i}{g\mathbb{Z}}\right| - n + 1 \geq g - n + 1$ (since $\cup_{i=1}^{n}(A_i + \tau_i) \sim \mathbb{N}$); so $\left|\frac{(h_1+\cdots+h_n)(A\setminus X)}{g\mathbb{Z}}\right| \geq g - n + 1$. Next, from the nature of the sequence $\left(\left|\frac{r(A\setminus X)}{g\mathbb{Z}}\right|\right)_{r\in\mathbb{N}}$ (pointed out in Lemma 3.3 of the next section) and the hypothesis that $A \setminus X$ is a basis, we derive that $\left|\frac{(h_1+\cdots+h_n+n)(A\setminus X)}{g\mathbb{Z}}\right| = g$; hence $\frac{(h_1+\cdots+h_n+n)(A\setminus X)}{g\mathbb{Z}} = \frac{\mathbb{Z}}{g\mathbb{Z}}$. We thus have $((h_1 + \cdots + h_n + n)(A \setminus X))^{(g)} \sim \mathbb{N}$. But since on the other hand we have (in view of the elementary properties of §2.1): $((h_1 + \cdots + h_n + n)(A \setminus X))^{(g)} = ((A_0 + \cdots + A_n) + (n - h_0)(A \setminus X))^{(g)} \sim (A_0 + \cdots + A_n)^{(g)} + (n - h_0)(A \setminus X) \sim A_0 + \cdots + A_n + (n - h_0)(A \setminus X) = (h_1 + \cdots + h_n + n)(A \setminus X)$, it finally follows that $(h_1 + \cdots + h_n + n)(A \setminus X) \sim \mathbb{N}$, that is $\mathrm{G}(A \setminus X) \leq h_1 + \cdots + h_n + n$.

In the work of Nash [7], the parameter n depends on h and $|X|$. Actually, its dependence in $|X|$ stems from the upper bounds of the cardinalities of the sets ℓX $(\ell = 0, \dots, h)$. In [7], the upper bound used for each $|\ell X|$ is $\binom{|X|+\ell-1}{\ell}$, which is a polynomial in ℓ with degree $(|X| - 1)$ and then leads to bound from above $\mathrm{G}(A \setminus X)$ by a polynomial in h with degree $(|X| + 1)$. However, that estimate of $|\ell X|$ is very large for many sets X; for example if X is an arithmetic progression, we simply have $|\ell X| = \ell|X| - \ell + 1$ which is linear in ℓ and (as we will see it later) allows to estimate $\mathrm{G}(A \setminus X)$ by a polynomial with degree 3 in h. In order to obtain such an estimate for $\mathrm{G}(A \setminus X)$ in the general case, our idea (see Lemmas 3.1 and 3.2) consists to replace $|X|$ by another parameter in X (resp. X and A) for which the cardinality of each of the sets ℓX (resp. other more complex sets) is bounded above by a linear function in ℓ (resp. simple function in h). The upper bounds obtained in this way for $\mathrm{G}(A \setminus X)$ are simply polynomials in h with degrees 3 or 2 and with coefficients linear in the considered parameters (see Theorems 4.1 and 4.3). On the other hand, it must be noted that upper bounds for $\mathrm{G}(A \setminus X)$ which are polynomials with degrees 3 or 2 in h can be directly derived from the theorem of Nash, but in this way we lose the linearity in the considered parameter (see Theorem 4.4 and Remark 4.5).

3 Lemmas

The two first lemmas which follow constitute the main differences with Nash' work [7] about the use of Kneser's theorems. While the third one gives the nature (in terms of monotony) of some sequences (related to a given finite abelian group) which also plays a vital part in the proof of our results.

Lemma 3.1 *Let X be a nonempty finite set of integers. Then we have:*

$$|X| \leq \frac{\mathrm{diam}(X)}{\delta(X)} + 1.$$

In addition, this inequality becomes an equality if and only if X is an arithmetic progression.

Proof. The lemma is obvious if $|X| = 1$. Assume for the following that $|X| \geq 2$ and write $X = \{x_0, x_1, \dots, x_n\}$ $(n \geq 1)$, with $x_0 < x_1 < \cdots < x_n$. Since the positive integers $x_i - x_{i-1}$ $(i = 1, \dots, n)$ are clearly multiples of $\delta(X)$ then we have $x_i - x_{i-1} \geq \delta(X)$ $(\forall i = 1, \dots, n)$. It follows that $x_n - x_0 = \sum_{i=1}^{n}(x_i - x_{i-1}) \geq n\delta(X)$, which gives $n \leq \frac{x_n - x_0}{\delta(X)} = \frac{\mathrm{diam}(X)}{\delta(X)}$. Hence $|X| = n + 1 \leq \frac{\mathrm{diam}(X)}{\delta(X)} + 1$ as required.
Further, the above proof shows well that the inequality of the lemma is reached if and only if we have $x_i - x_{i-1} = \delta(X)$ $(\forall i = 1, \dots, n)$ which simply means that X is an arithmetic progression. The proof is complete. ■

Lemma 3.2 *Let X be a finite nonempty set of integers and B be an infinite set of integers having a finite intersection with $\mathbb{Z}^-$. Define:*

$$\eta := \min_{\substack{b,b' \in B, b \neq b' \\ |b - b'| \geq \mathrm{diam}(X)}} |b - b'|.$$

Then, for all $u, v \in \mathbb{N}$, $g \in \mathbb{N}^*$, we have:

$$(uB + vX)(m) \leq \eta.((u+v)B)(m) + O(1)$$

and

$$\left|\frac{uB + vX}{g\mathbb{Z}}\right| \leq \eta \left|\frac{(u+v)B}{g\mathbb{Z}}\right|.$$

Proof. Since we have for all $\tau \in \mathbb{Z}$: $(uB + vX + \tau)(m) = (uB + vX)(m) + O(1)$ (according to the part (6) of §2.1) and $\left|\frac{uB+vX+\tau}{g\mathbb{Z}}\right| = \left|\frac{uB+vX}{g\mathbb{Z}}\right|$ (obviously), then there is no loss of generality in translating B and X by integers. By translating, if necessary, X, assume that 0 is its smaller element and write $X = \{x_0, x_1, \dots, x_n\}$ $(n \in \mathbb{N})$, with $0 = x_0 < x_1 < \cdots < x_n$. Next, let $b_0, b \in B$ such that $b - b_0 = \eta$. By translating, if necessary, B, assume $b_0 = 0$. Then we have

$$b = \eta \geq \mathrm{diam}(X) = x_n.$$

In this situation, we claim that we have

$$(uB + vX) \subset \bigcup_{0 \leq \tau < \eta} ((u+v)B + \tau) \tag{1}$$

which clearly implies the two inequalities of the lemma. So, it just remains to show (1). Let $N \in (uB + vX)$ and show that there exists a non-negative integer $\tau < \eta$ such that $N \in (u+v)B + \tau$. Since $0 = b_0 = x_0 \in B \cap X$, the fact that $N \in (uB + vX)$ means that N can be written in the form

$$N = u_1 b_1 + \cdots + u_k b_k + v_1 x_1 + \cdots + v_n x_n, \tag{2}$$

with $k, u_1, \dots, u_k, v_1, \dots, v_n \in \mathbb{N}$, $b_1, \dots, b_k \in B$, $u_1 + \cdots + u_k \leq u$ and $v_1 + \cdots + v_n \leq v$.
Now, since $x_1 < x_2 < \cdots < x_n \leq \eta$, then we have $v_1 x_1 + \cdots + v_n x_n \leq (v_1 + \cdots + v_n)\eta \leq v\eta$, which implies that the euclidean division of the non-negative integer $(v_1 x_1 + \cdots + v_n x_n)$ by η yields:

$$v_1 x_1 + \cdots + v_n x_n = t\eta + \tau, \tag{3}$$

with $t, \tau \in \mathbb{N}$, $t \leq v$ and $0 \leq \tau < \eta$. By reporting (3) into (2), we finally obtain

$$N = u_1 b_1 + \cdots + u_k b_k + t\eta + \tau. \tag{4}$$

Since $0 = b_0 \in B$, $b_1, \dots, b_k, \eta \in B$ (recall that $\eta = b$) and $u_1 + \cdots + u_k + t \leq u + v$, then the relation (4) is well a writing of N as a sum of $(u + v)$ elements of B and τ; in other words $N \in (u + v)B + \tau$, giving the desired conclusion. The proof is complete. ■

Lemma 3.3 *Let G be a finite abelian group and B be a nonempty subset of G. For all $r \in \mathbb{N}$, set $u_r := |rB|$. Then, there exists $r_0 \in \mathbb{N}$ such that:*

$$u_0 < u_1 < \cdots < u_{r_0}$$

and

$$u_r = u_{r_0} \qquad (\forall r \geq r_0).$$

Proof. Firstly, since G is finite, the sequence $(u_r)_r$ is bounded above by $|G|$. Secondly, we claim that $(u_r)_r$ is nondecreasing. Indeed, by fixing $b \in B$, we have for all $r \in \mathbb{N}$: $(r + 1)B \supset rB + b$, hence $u_{r+1} = |(r + 1)B| \geq |rB + b| = |rB| = u_r$. It follows from these two facts that there exists $r_0 \in \mathbb{N}$ such that $u_{r_0} = u_{r_0+1}$. By taking r_0 minimal to have this property, we have:

$$u_0 < u_1 < \cdots < u_{r_0} = u_{r_0+1}.$$

To conclude the proof of the lemma, it remains to show that

$$u_r = u_{r_0} \qquad (\forall r \geq r_0). \tag{5}$$

If $b \in B$ is fixed, we claim that for all $r \geq r_0$, we have:

$$rB = r_0 B + (r - r_0)b \tag{6}$$

which clearly implies (5). So, it remains to show (6). To do this, we argue by induction on $r \geq r_0$. For $r = r_0$, the relation (6) is obvious. Next, since $(r_0+1)B \supset r_0 B + b$ and $|(r_0 + 1)B| = u_{r_0+1} = u_{r_0} = |r_0 B| = |r_0 B + b|$, then we certainly have $(r_0 + 1)B = r_0 B + b$, showing that (6) also holds for $r = r_0 + 1$. Now, let $r \geq r_0$, assume that (6) holds for r and show that it also holds for

$(r+1)$. We have:

$$\begin{aligned}
(r+1)B &= (r_0+1)B + (r-r_0)B \\
&= (r_0B+b) + (r-r_0)B \quad \text{(since (6) holds for } (r_0+1)) \\
&= rB + b \\
&= (r_0B + (r-r_0)b) + b \quad \text{(from the induction hypothesis)} \\
&= r_0B + (r+1-r_0)b.
\end{aligned}$$

Hence (6) also holds for $(r+1)$. This finishes this induction and completes the proof. ■

4 Main Results

Throughout this section, we fix an additive basis A and a finite nonempty subset X of A such that $A \setminus X$ is still a basis. We put $h := \mathrm{G}(A)$ and we define

$$d := \frac{\mathrm{diam}(X)}{\delta(X)} \ , \ \eta := \min_{\substack{a,b \in A\setminus X, a\neq b \\ |a-b| \geq \mathrm{diam}(X)}} |a-b| \ \text{ and } \ \mu := \min_{y \in A\setminus X} \mathrm{diam}(X \cup \{y\}).$$

Theorem 4.1 *We have* $\mathrm{G}(A \setminus X) \leq \frac{h(h+3)}{2} + d\frac{h(h-1)(h+4)}{6}$.

Proof. Put $B := A \setminus X$, so $A = B \cup X$. Then, the fact that A is a basis of order h amounts to:

$$hB \cup ((h-1)B + X) \cup ((h-2)B + 2X) \cup \cdots \cup (B + (h-1)X) \sim \mathbb{N}. \qquad (7)$$

(Remark that hX is finite).
Now, since the set of the left-hand side of (7) is clearly contained in a finite union of translates of hB, then by denoting N a number of translates of hB which are sufficient to cover it, we have (according to the part (6) of §2.1):

$$(hB \cup ((h-1)B + X) \cup \cdots \cup (B + (h-1)X))\,(m) \leq N.(hB)(m) + O(1).$$

It follows that:

$$\begin{aligned}
&\liminf_{m\to+\infty} \frac{(hB)(m)}{m} \\
&\quad \geq \frac{1}{N} \liminf_{m\to+\infty} \frac{1}{m}\,(hB \cup ((h-1)B + X) \cup \cdots \cup (B + (h-1)X))\,(m) \\
&\quad = \frac{1}{N} \qquad \text{(according to (7))}.
\end{aligned}$$

Thus

$$\underline{\mathbf{d}}(hB) \geq \frac{1}{N} > 0. \tag{8}$$

Now, according to (7), (8) and the part (5) of §2.1, we have:

$$\begin{aligned}
&\underline{\mathbf{d}}\left(hB \vee hB \vee ((h-1)B+X) \vee ((h-2)B+2X) \vee \cdots \vee (B+(h-1)X)\right) \\
&\quad \geq \underline{\mathbf{d}}(hB) + \underline{\mathbf{d}}\left(hB \vee ((h-1)B+X) \vee \cdots \vee (B+(h-1)X)\right) \\
&\quad \geq \underline{\mathbf{d}}(hB) + \underline{\mathbf{d}}\left(hB \cup ((h-1)B+X) \cup \cdots \cup (B+(h-1)X)\right) \\
&\quad = \underline{\mathbf{d}}(hB) + 1 \\
&\quad > 1.
\end{aligned}$$

So, we have

$$\begin{aligned}
\liminf_{m \to +\infty} \frac{1}{m} \{&(hB)(m) + (hB)(m) + ((h-1)B+X)(m) \\
&+ ((h-2)B+2X)(m) + \cdots + (B+(h-1)X)(m)\} > 1.
\end{aligned} \tag{9}$$

Next, according to the part (6) of §2.1 and to Lemma 3.1, each of the quantities $((h-\ell)B+\ell X)(m)$ $(\ell = 1, \ldots, h-1)$ is bounded above as follows

$$\begin{aligned}
((h-\ell)B+\ell X)(m) &\leq |\ell X|.((h-\ell)B)(m) + O(1) \\
&\leq \left(\frac{\mathrm{diam}(\ell X)}{\delta(\ell X)} + 1\right).((h-\ell)B)(m) + O(1) \\
&= (\ell d+1).((h-\ell)B)(m) + O(1)
\end{aligned} \tag{10}$$

(since $\mathrm{diam}(\ell X) = \ell \mathrm{diam}(X)$ and $\delta(\ell X) = \delta(X)$).
Then, by reporting these into (9), we obtain:

$$\begin{aligned}
\liminf_{m \to +\infty} \frac{1}{m} \{&(hB)(m) + (hB)(m) + (d+1).((h-1)B)(m) \\
&+ (2d+1).((h-2)B)(m) + \cdots + ((h-1)d+1).B(m)\} > 1,
\end{aligned}$$

which amounts to

$$\underline{\mathbf{d}}\left(hB \vee \bigvee_{\ell=0}^{h-1}\left(\bigvee_{(\ell d+1)\text{ times}} (h-\ell)B\right)\right) > 1. \tag{11}$$

This last relation shows well that the first alternative of the first theorem of Kneser (applied to the set hB with $(\ell d+1)$ copies of each of the sets $(h-\ell)B$, $\ell = 0, 1, \ldots, h-1$) cannot hold. We are thus in the second alternative of the first theorem of Kneser; that is there exists a positive integer g such that

$$\left(h + \sum_{\ell=0}^{h-1}(\ell d+1)(h-\ell)\right) B \sim \left(\left(h + \sum_{\ell=0}^{h-1}(\ell d+1)(h-\ell)\right) B\right)^{(g)}. \tag{12}$$

Let's take g minimal in (12). This implies from Proposition 2.5 that the set $(h+\sum_{\ell=0}^{h-1}(\ell d+1)(h-\ell))B$ is not degenerate modulo g; in other words, the set $(h+\sum_{\ell=0}^{h-1}(\ell d+1)(h-\ell))\frac{B}{g\mathbb{Z}}$ is not degenerate in $\frac{\mathbb{Z}}{g\mathbb{Z}}$. It follows from Proposition 2.3 that also the set $(\sum_{\ell=0}^{h-1}(\ell d+1)(h-\ell))\frac{B}{g\mathbb{Z}}$ is not degenerate in $\frac{\mathbb{Z}}{g\mathbb{Z}}$. Then, from Corollary 2.4, we have

$$\begin{aligned} \left|\left(\sum_{\ell=0}^{h-1}(\ell d+1)(h-\ell)\right)\frac{B}{g\mathbb{Z}}\right| &= \left|\sum_{\ell=0}^{h-1}\sum_{(\ell d+1)\text{ times}}\frac{(h-\ell)B}{g\mathbb{Z}}\right| \\ &\geq \sum_{\ell=0}^{h-1}(\ell d+1)\left|\frac{(h-\ell)B}{g\mathbb{Z}}\right| - \sum_{\ell=0}^{h-1}(\ell d+1)+1. \end{aligned} \tag{13}$$

Now, let's bound from below the sum $\sum_{\ell=0}^{h-1}(\ell d+1)\left|\frac{(h-\ell)B}{g\mathbb{Z}}\right|$. We have for all $\ell\in\{0,1,\dots,h-1\}$:

$$\begin{aligned} (\ell d+1)\left|\frac{(h-\ell)B}{g\mathbb{Z}}\right| &= \left(\frac{\operatorname{diam}(\ell X)}{\delta(\ell X)}+1\right)\left|\frac{(h-\ell)B}{g\mathbb{Z}}\right| \\ &\geq |\ell X|.\left|\frac{(h-\ell)B}{g\mathbb{Z}}\right| \qquad \text{(according to Lemma 3.1)} \\ &\geq \left|\frac{\ell X}{g\mathbb{Z}}\right|.\left|\frac{(h-\ell)B}{g\mathbb{Z}}\right| \\ &\geq \left|\frac{(h-\ell)B+\ell X}{g\mathbb{Z}}\right|; \end{aligned}$$

hence

$$\begin{aligned} \sum_{\ell=0}^{h-1}(\ell d+1)\left|\frac{(h-\ell)B}{g\mathbb{Z}}\right| &\geq \sum_{\ell=0}^{h-1}\left|\frac{(h-\ell)B+\ell X}{g\mathbb{Z}}\right| \\ &\geq \left|\frac{hB\cup((h-1)B+X)\cup\cdots\cup(B+(h-1)X)}{g\mathbb{Z}}\right| \\ &= g \qquad \text{(according to (7)).} \end{aligned}$$

By reporting this into (13), we have

$$\left|\left(\sum_{\ell=0}^{h-1}(\ell d+1)(h-\ell)\right)\frac{B}{g\mathbb{Z}}\right| \geq g-\sum_{\ell=0}^{h-1}(\ell d+1)+1. \tag{14}$$

Now, from Lemma 3.3, we know that the sequence of natural numbers $\left(\left|r\frac{B}{g\mathbb{Z}}\right|\right)_{r\in\mathbb{N}}$ increases until reaching its maximal value which it then continues to take indefinitely. In addition, because $\mathrm{G}(B)B\sim\mathbb{N}$, we have $\left|\mathrm{G}(B)\frac{B}{g\mathbb{Z}}\right|=\left|\frac{\mathbb{Z}}{g\mathbb{Z}}\right|=g$, showing that g is the maximal value of the same sequence. On the other hand, if we assume that the finite sequence

$\left(\left|r\frac{B}{g\mathbb{Z}}\right|\right)_{\sum_{\ell=0}^{h-1}(\ell d+1)(h-\ell)\leq r\leq\sum_{\ell=0}^{h-1}(\ell d+1)(h-\ell+1)}$ is increasing, we would have (according to (14)):

$$\left|\left(\sum_{\ell=0}^{h-1}(\ell d+1)(h-\ell+1)\right)\frac{B}{g\mathbb{Z}}\right| \geq g+1$$

which is impossible. Consequently, the sequence $\left(\left|r\frac{B}{g\mathbb{Z}}\right|\right)_{r\in\mathbb{N}}$ becomes constant (equal to g) before its term of order $r = \sum_{\ell=0}^{h-1}(\ell d+1)(h-\ell+1)$. In particular, we have

$$\left|\left(\sum_{\ell=0}^{h-1}(\ell d+1)(h-\ell+1)\right)\frac{B}{g\mathbb{Z}}\right| = g$$

and then

$$\left(\sum_{\ell=0}^{h-1}(\ell d+1)(h-\ell+1)\right)\frac{B}{g\mathbb{Z}} = \frac{\mathbb{Z}}{g\mathbb{Z}},$$

implying that

$$\left(\left(\sum_{\ell=0}^{h-1}(\ell d+1)(h-\ell+1)\right)B\right)^{(g)} = \mathbb{N}. \tag{15}$$

But on the other hand, since $\sum_{\ell=0}^{h-1}(\ell d+1)(h-\ell+1) \geq h+\sum_{\ell=0}^{h-1}(\ell d+1)(h-\ell)$, we have (according to the relation (12) and the property of the part (7) of §2.1):

$$\left(\sum_{\ell=0}^{h-1}(\ell d+1)(h-\ell+1)\right)B \sim \left(\left(\sum_{\ell=0}^{h-1}(\ell d+1)(h-\ell+1)\right)B\right)^{(g)}. \tag{16}$$

By comparing (15) and (16), we finally deduce that

$$\left(\sum_{\ell=0}^{h-1}(\ell d+1)(h-\ell+1)\right)B \sim \mathbb{N},$$

which gives

$$\mathrm{G}(B) \leq \sum_{\ell=0}^{h-1}(\ell d+1)(h-\ell+1) = \frac{h(h+3)}{2} + d\frac{h(h-1)(h+4)}{6}$$

(since $\sum_{\ell=0}^{h-1}\ell = \frac{h(h-1)}{2}$ and $\sum_{\ell=0}^{h-1}\ell^2 = \frac{h(h-1)(2h-1)}{6}$).
The theorem is proved. ■

Corollary 4.2 *If in addition X is an arithmetic progression, then we have:*

$$\mathrm{G}(A\setminus X) \leq \frac{h(h+3)}{2} + (|X|-1)\frac{h(h-1)(h+4)}{6}.$$

Proof. By Lemma 3.1, we have $|X| = \frac{\mathrm{diam}(X)}{\delta(X)}+1 = d+1$, hence $d = |X|-1$. The corollary then follows at once from Theorem 4.1. ■

Theorem 4.3 *We have* $\mathrm{G}(A \setminus X) \leq \eta(h^2-1)+h+1$.

Proof. We proceed as in the proof of Theorem 4.1 with some differences; so we only detail these differences. Putting $B := A \setminus X$, we repeat the proof of Theorem 4.1 until the relation (9). After that, using Lemma 3.2, we bound from above each of the quantities $((h-\ell)B+\ell X)(m)$ $(\ell = 1, \dots, h-1)$ by

$$((h-\ell)B+\ell X)(m) \leq \eta.(hB)(m) + O(1). \tag{10'}$$

Then, by reporting these into (9), we obtain

$$\underline{\mathbf{d}}\left(\bigvee_{(\eta(h-1)+2)\ \text{times}} (hB)\right) > 1, \tag{11'}$$

which shows well that the first alternative of the first theorem of Kneser (applied to $(\eta(h-1)+2)$ copies of the set hB) cannot hold. Consequently, we are in the second alternative of the first theorem of Kneser, that is there exists a positive integer g such that

$$(\eta(h-1)+2)hB \sim \left((\eta(h-1)+2)hB\right)^{(g)}. \tag{12'}$$

Let's take g minimal in $(12')$. Then, Propositions 2.5 and 2.3 imply that the set $(\eta(h-1)+1)h\frac{B}{g\mathbb{Z}}$ is non degenerate in $\frac{\mathbb{Z}}{g\mathbb{Z}}$. It follows from Corollary 2.4 that we have

$$\begin{aligned}\left|(\eta(h-1)+1)h\frac{B}{g\mathbb{Z}}\right| &= \left|\sum_{(\eta(h-1)+1)\ \text{times}} \frac{hB}{g\mathbb{Z}}\right| \\ &\geq (\eta(h-1)+1)\left|\frac{hB}{g\mathbb{Z}}\right| - \eta(h-1). \end{aligned} \tag{13'}$$

Next, using the second inequality of Lemma 3.2, we have

$$\begin{aligned}(\eta(h-1)+1)\left|\frac{hB}{g\mathbb{Z}}\right| &= \sum_{\ell=1}^{h-1} \eta.\left|\frac{((h-\ell)+\ell)B}{g\mathbb{Z}}\right| + \left|\frac{hB}{g\mathbb{Z}}\right| \\ &\geq \sum_{\ell=1}^{h-1}\left|\frac{(h-\ell)B+\ell X}{g\mathbb{Z}}\right| + \left|\frac{hB}{g\mathbb{Z}}\right| \\ &\geq \left|\bigcup_{\ell=0}^{h-1}\frac{((h-\ell)B+\ell X)}{g\mathbb{Z}}\right| \\ &= g \qquad \text{(according to (7)).}\end{aligned}$$

By reporting this into $(13')$, we have

$$\left|(\eta(h-1)+1)h\frac{B}{g\mathbb{Z}}\right| \geq g - \eta(h-1). \tag{14'}$$

It follows from Lemma 3.3 (as we applied it in the proof of Theorem 4.1) that the sequence $\left(\left|r\frac{B}{g\mathbb{Z}}\right|\right)_{r\in\mathbb{N}}$ is stationary in g before its term of order $r = (\eta(h-1)+1)(h+1)$. In particular, we have $\left|(\eta(h-1)+1)(h+1)\frac{B}{g\mathbb{Z}}\right| = g$; hence $(\eta(h-1)+1)(h+1)\frac{B}{g\mathbb{Z}} = \frac{\mathbb{Z}}{g\mathbb{Z}}$, implying that

$$((\eta(h-1)+1)(h+1)B)^{(g)} \sim \mathbb{N}. \tag{15'}$$

But on the other hand, since $\eta \geq 1$, we have $(\eta(h-1)+1)(h+1) \geq (\eta(h-1)+2)h$, which implies (according to the relation (12′) and the property of the part (7) of §2.1) that

$$(\eta(h-1)+1)(h+1)B \sim ((\eta(h-1)+1)(h+1)B)^{(g)}. \tag{16'}$$

By comparing (15′) and (16′), we finally deduce that

$$(\eta(h-1)+1)(h+1)B \sim \mathbb{N},$$

which gives $\mathrm{G}(B) \leq (\eta(h-1)+1)(h+1) = \eta(h^2-1)+h+1$, as required. The theorem is proved. ■

Theorem 4.4 *We have* $\mathrm{G}(A \setminus X) \leq \dfrac{h\mu(h\mu+3)}{2}$.

Proof. First, notice that $\mu \geq 1$ (since $X \neq \emptyset$). Notice also that the parameters h, μ and $G(A \setminus X)$ are still unchanged if we translate the basis A by an integer. Let $y_0 \in A \setminus X$ such that $\mu = \mathrm{diam}(X \cup \{y_0\})$; so by translating if necessary A by $(-y_0)$, we can assume (without loss of generality) that $y_0 = 0$. Then putting $X = \{x_1, \dots, x_n\}$ $(n \geq 1)$ with $x_1 < x_2 < \cdots < x_n$, we have

$$\mu = \mathrm{diam}(X \cup \{0\}) = \max\{|x_1|, |x_2|, \dots, |x_n|, x_n - x_1\}. \tag{17}$$

We are going to show that the set $(A \setminus X) \cup \{\pm 1\}$ is a basis of order $\leq h\mu$. The result of the theorem then follows from the particular case '$k = 1$' of Theorem 1.1 of Nash. We distinguish the three following cases:

1st case. (if $x_1 \geq 0$)
In this case, the elements of X are all non-negative. Let N be a natural number large enough that it can be written as a sum of h elements of A; that is

$$N = a_1 + \cdots + a_t + \alpha_1 x_1 + \cdots + \alpha_n x_n, \tag{18}$$

with $t, \alpha_1, \dots, \alpha_n \in \mathbb{N}$, $a_1, \dots, a_t \in A \setminus X$ and $t + \alpha_1 + \cdots + \alpha_n = h$.
Next, since the non-negative integer $(\alpha_1 x_1 + \cdots + \alpha_n x_n)$ is obviously bounded above by $(\alpha_1 + \cdots + \alpha_n)\mu = (h-t)\mu \leq h\mu - t$, then it is a sum of $(h\mu - t)$ elements of the set $\{0, 1\}$. It follows from (18) that N is a sum of $h\mu$ elements of the set $(A \setminus X) \cup \{0, 1\} = (A \setminus X) \cup \{1\}$. This last fact shows well (since N is an arbitrary sufficiently large integer) that the set $(A \setminus X) \cup \{1\}$ is a basis of

order $h' \leq h\mu$. Hence

• either $1 \in A \setminus X$, in which case we have $(A \setminus X) = (A \setminus X) \cup \{1\}$ and then $G(A \setminus X) = h' \leq h\mu \leq \frac{h\mu(h\mu+3)}{2}$,

• or $1 \notin A \setminus X$, in which case we have $(A \setminus X) = ((A \setminus X) \cup \{1\}) \setminus \{1\}$, implying (according to Theorem 1.1 for $k = 1$) that $\mathrm{G}(A \setminus X) \leq \frac{h'(h'+3)}{2} \leq \frac{h\mu(h\mu+3)}{2}$.

So, in this first case, we always have $\mathrm{G}(A \setminus X) \leq \frac{h\mu(h\mu+3)}{2}$ as required.

2^nd case. (if $x_n \leq 0$)

In this case, the elements of X are all non-positive. Let N be a natural number large enough that can be written as a sum of h elements of A; that is

$$N = a_1 + \cdots + a_t + \alpha_1 x_1 + \cdots + \alpha_n x_n, \tag{19}$$

with $t, \alpha_1, \ldots, \alpha_n \in \mathbb{N}$, $a_1, \ldots, a_t \in A \setminus X$ and $t + \alpha_1 + \cdots + \alpha_n = h$.

Next, since the non-positive integer $(\alpha_1 x_1 + \cdots + \alpha_n x_n)$ is bounded below by $-(\alpha_1 + \cdots + \alpha_n)\mu = (t - h)\mu \geq t - h\mu$, then it is a sum of $(h\mu - t)$ elements of the set $\{0, -1\}$. It follows from (19) that N is a sum of $h\mu$ elements of the set $(A \setminus X) \cup \{0, -1\} = (A \setminus X) \cup \{-1\}$. This shows well (since N is an arbitrary sufficiently large integer) that the set $(A \setminus X) \cup \{-1\}$ is a basis of order $\leq h\mu$. We finally conclude (like in the first case) that $\mathrm{G}(A \setminus X) \leq \frac{h\mu(h\mu+3)}{2}$ as required.

3^rd case. (if $x_1 < 0$ and $x_n > 0$)

In this case, we have (from (17)) that $\mu = x_n - x_1$. Let N be a natural number large enough so that the number $(N + hx_1)$ can be written as a sum of h elements of A; that is

$$N + hx_1 = a_1 + \cdots + a_t + \alpha_1 x_1 + \cdots + \alpha_n x_n, \tag{20}$$

with $t, \alpha_1, \ldots, \alpha_n \in \mathbb{N}$, $a_1, \ldots, a_t \in A \setminus X$ and $t + \alpha_1 + \cdots + \alpha_n = h$.

From the identity

$$\alpha_1 x_1 + \cdots + \alpha_n x_n - hx_1 = \alpha_2(x_2 - x_1) + \alpha_3(x_3 - x_1) + \cdots + \alpha_n(x_n - x_1) - tx_1,$$

we deduce (since $0 < x_2 - x_1 < x_3 - x_1 < \cdots < x_n - x_1 = \mu$ and $0 < -x_1 \leq x_n - x_1 - 1 = \mu - 1$) that

$$0 < \alpha_1 x_1 + \cdots + \alpha_n x_n - hx_1 \leq (\alpha_2 + \cdots + \alpha_n)\mu + t(\mu - 1) \leq h\mu - t,$$

which implies that the integer $(\alpha_1 x_1 + \cdots + \alpha_n x_n - hx_1)$ can be written as a sum of $(h\mu - t)$ elements of the set $\{0, 1\}$. It follows from (20) that N is a sum of $h\mu$ elements of the set $(A \setminus X) \cup \{0, 1\} = (A \setminus X) \cup \{1\}$. This shows that the set $(A \setminus X) \cup \{1\}$ is a basis of order $\leq h\mu$ and leads (as in the first case) to the desired estimate $\mathrm{G}(A \setminus X) \leq \frac{h\mu(h\mu+3)}{2}$. The proof is complete. ∎

Remark 4.5 *By using Theorem 1.1 of Nash for $k = 1, 2$, we can also establish by an elementary way (like in the above proof of Theorem 4.4) an upper bound for* $\mathrm{G}(A \setminus X)$ *in function of h and d. Actually, we obtain*

$$\mathrm{G}(A \setminus X) \leq \frac{hd(hd+1)(hd+5)}{6}.$$

But this estimate is weaker than that of Theorem 4.1 and in addition it is not linear in d.

Some open questions:

(1) Does there exist an upper bound for $\mathrm{G}(A\setminus X)$, depending only on h and d, which is polynomial in h with degree 2 and linear in d? (This asks about the improvement of Theorem 4.1).

(2) Does there exist an upper bound for $\mathrm{G}(A\setminus X)$, depending only on h and μ, which is polynomial in h with degree 2 and linear in μ? (This asks about the improvement of Theorem 4.4).

References

[1] P. Erdös & R. L. Graham. On bases with an exact order, *Acta Arith*, **37** (1980), p. 201-207.

[2] G. Grekos. Quelques aspects de la Théorie Additive des Nombres, Thèse, Université de Bordeaux I, juin 1982.

[3] —— Sur l'ordre d'une base additive, *séminaire de théorie des nombres de Bordeaux*, exposé 31, année 1987/88.

[4] H. Halberstam & K. Roth. Sequences, *Oxford University Press*, (1966).

[5] M. Kneser. Abschätzung der asymptotischen Dichte von Summenmengen, *Math. Z*, **58** (1953), p. 459-484.

[6] J. C. M. Nash. Results in Bases in Additive Number Theory, Thesis, Rutgers University, New Jersey, 1985.

[7] —— Some applications of a theorem of M. Kneser, *J. Number Theory*, **44** (1993), p. 1-8.

[8] J. C. M. Nash & M. B. Nathanson. Cofinite subsets of asymptotic bases for the positive integers, *J. Number Theory*, **20** (1985), p. 363-372.

[9] M. B. Nathanson. The exact order of subsets of additive bases, in "Proceedings, Number Theory Seminar, 1982," Lecture Notes in Mathematics, Vol. 1052, p. 273-277, Springer-Verlag, 1984.

[10] A. Plagne. À propos de la fonction X d'Erdös et Graham, *Ann. Inst. Fourier*, **54**, 6 (2004), p. 1-51.

[11] Xing-de Jia. Exact Order of Subsets of Asymptotic Bases in Additive Number Theory, *J. Number Theory*, **28** (1988), p. 205-218.